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CALCULATION OF THE OUTFLOW OF A ONE-VELOCITY HETEROGENEOUS MIXTURE INTO VACUUM

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The outflow of a water-air mixture (an incompressible fluid) and a mixture of two ideal gases has been studied within the framework of the generalized-equilibrium model of a heterogeneous medium. In the first problem, the solution has been obtained in the form of analytical dependences. In the other, for a complete determination of all the dependent variables both analytical relations and an ordinary differential equation have been integrated numerically. The outflow of a mixture with an arbitrary number of components into vacuum is calculated similarly.

The available one-velocity models of a dispersed medium do not allow adequate description of a number of phenomena in foamy and bubble fluids; therefore, the so-called generalized-equilibrium model of a heterogeneous medium has been suggested [1]. In this model, along with the equations of mass, momentum, and energy conservation for the mixture as a whole, use is made of the corresponding relations for individual components; therefore, the total number of formulas in the model is proportional to the number of fractions in the mixture. The systems of equations appear to be much more complex compared to those used traditionally. In the generalized-equilibrium model, the properties of the mixture are determined by the constants of individual substances which make up the mixture; in this case, there are no additional parameters which are introduced in a number of models for coordination of calculated and experimental data. In [1–3], numerical methods of calculation of complete systems of equations of the model are suggested, which is connected with a rather labor-consuming procedure of integration of partial differential equations. For a number of problems, the solution can be obtained by simpler means, which, in particular, is studied in the present work.

We first consider the outflow of a binary mixture with one incompressible component. Let the half-space $r < 0$, filled with a homogeneous dispersed medium with a density ρ_0 and a volume fraction of a gas in the mixture α_0 and being under pressure p_0, be separated from the vacuum region ($p = 0$, $\rho = 0$) by an impermeable orifice lying at the point $r = 0$. It is necessary to calculate the flow which arises in instantaneous (at the instant of time $t = 0$) destruction of the orifice. The determining equations have the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = 0\,, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0\,,$$
$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \frac{\gamma p}{\alpha}\frac{\partial u}{\partial r} = 0\,, \quad \frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial r} - (1-\alpha)\frac{\partial u}{\partial r} = 0\,, \qquad (1)$$

where α is the volume fraction of the gas in the mixture and $\rho = \alpha\rho_1^0 + (1-\alpha)\rho_2^0$. The first three equations in (1) are the laws of conservation of mass, momentum, and energy for the mixture as a whole, while the last equation is the continuity equation for an incompressible fraction [1].

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By introduction of the self-similar variable $\xi = r/t$ the system of partial differential equations (1) with account for the relations $\partial/\partial t = -(r/t^2)(d/d\xi)$ and $\partial/\partial r = (1/t)(d/d\xi)$ is reduced to the system of ordinary differential equations

$$(u-\xi)\frac{d\rho}{d\xi}+\rho\frac{du}{d\xi}=0\,,\quad (u-\xi)\frac{du}{d\xi}+\frac{1}{\rho}\frac{dp}{d\xi}=0\,,$$
$$(u-\xi)\frac{dp}{d\xi}+\frac{\gamma p}{\alpha}\frac{du}{d\xi}=0\,,\quad (u-\xi)\frac{d\alpha}{d\xi}-(1-\alpha)\frac{du}{d\xi}=0\,. \tag{2}$$

We note that the equation for the entropy $\partial S/\partial t + u(\partial S/\partial r) = 0$ [1] is transformed to $(u-\xi)(dS/d\xi) = 0$, whence it is seen that $dS = 0$ or $S = \text{const}$. Thus, the considered self-similar flow is isentropic.

Comparing the first and fourth relations in (2), we obtain the equation:

$$d\rho/\rho + d\alpha/(1-\alpha) = 0\,,$$

after integration of which with account for the boundary condition $\alpha = \alpha_0$ at $\rho = \rho_0$ we find the relation between the volume fraction of the gas and the density of the mixture:

$$\alpha = 1-(1-\alpha_0)\,\rho/\rho_0\,. \tag{3}$$

From the second and third equations of (2) we have the following relation:

$$\frac{dp}{d\xi}\left((u-\xi)^2-\frac{\gamma p}{\alpha\rho}\right)=0\,.$$

Since $dp/d\xi \neq 0$, the expression

$$u-\xi=\sqrt{\gamma p/(\alpha\rho)}=c\,. \tag{4}$$

holds. Using relation (4), we rewrite the first equation of system (2) in the form

$$c\frac{d\rho}{d\xi}+\rho\frac{du}{d\xi}=0\,. \tag{5}$$

Allowing for the expression for the velocity of sound in the mixture [4]

$$c=\frac{c_0\rho_0}{\rho}\left(\frac{\alpha_0\rho}{\rho_0-(1-\alpha_0)\,\rho}\right)^{(\gamma+1)/2}\left(c_0=\sqrt{\frac{\gamma p_0}{\alpha_0\rho_0}}\right),$$

we can write Eq. (5) as follows:

$$du=-\frac{c_0\rho_0}{\rho^2}\left(\frac{\alpha_0\rho}{\rho_0-(1-\alpha_0)\,\rho}\right)^{(\gamma+1)/2}d\rho\,.$$

Integration of the last equality yields

$$u=-c_0\rho_0\int\limits_{\rho_0}^{\rho}\left(\frac{\alpha_0\rho}{\rho_0-(1-\alpha_0)\,\rho}\right)^{(\gamma+1)/2}\frac{d\rho}{\rho^2}=$$

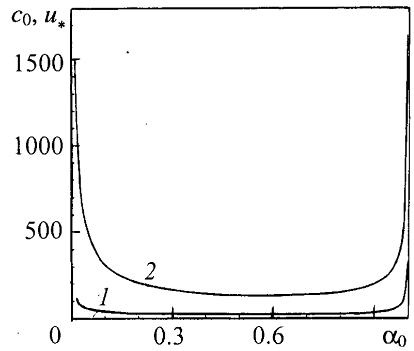

Fig. 1. Dependences $c_0(\alpha_0)$ and $u_*(\alpha_0)$ [1 and 2, respectively] for a water-air mixture with an incompressible liquid fraction.

$$= - c_0\alpha_0^{(\gamma+1)/2} \int\limits_{1/\alpha_0}^{\rho/(\rho_0-(1-\alpha_0)\rho)} t^{(\gamma-3)/2}\,dt = \frac{2c_0}{\gamma-1}\left[\alpha_0^{-(\gamma-1)/2} - \left(\frac{\rho}{\rho_0-(1-\alpha_0)\,\rho}\right)^{(\gamma-1)/2}\right].$$

In taking the integral, we made the substitution of the variable $t = \rho/[\rho_0 - (1 - \alpha_0)\rho]$. Inverting the obtained dependence $u = u(\rho)$, we have

$$\rho = \rho_0 \frac{\left(\alpha_0^{-(\gamma-1)/2} - \frac{(\gamma-1)\,u}{2c_0}\right)^{2/(\gamma-1)}}{1 + (1-\alpha_0)\left(\alpha_0^{-(\gamma-1)/2} - \frac{(\gamma-1)\,u}{2c_0}\right)^{2/(\gamma-1)}}. \qquad (6)$$

Similarly we can also obtain other dependences ($p = p(u)$, $\alpha = \alpha(u)$, and $c = c(u)$), which have the form

$$p = p_0\alpha_0^{\gamma}\left(\alpha_0^{-(\gamma-1)/2} - \frac{(\gamma-1)\,u}{2c_0}\right)^{2\gamma/(\gamma-1)},$$

$$\alpha = \left[1 + (1-\alpha_0)\left(\alpha_0^{-(\gamma-1)/2} - \frac{(\gamma-1)\,u}{2c_0}\right)^{2/(\gamma-1)}\right]^{-1}, \qquad (7)$$

$$c = \left(\alpha_0 c_0 - \frac{(\gamma-1)\,u\alpha_0^{(\gamma+1)/2}}{2c_0}\right)\left[1 + (1-\alpha_0)\left(\alpha_0^{-(\gamma-1)/2} - \frac{(\gamma-1)\,u}{2c_0}\right)^{2/(\gamma-1)}\right].$$

Relations (4), (6), and (7) determine the solution of the posed problem completely. It is seen from an analysis of the expressions presented that in a rarefaction wave the density of the medium and the pressure fall from the initial values to zero. The wave front propagates into vacuum at a constant velocity:

$$u_* = 2c_0\alpha_0^{-(\gamma-1)/2}/(\gamma-1).$$

The trailing edge of the rarefaction wave shifts to the depth of an undisturbed dispersed medium at the velocity of sound c_0. The dependences of the velocities of motion of the wave boundaries on the concentration of the gas in the water-air mixture are given in Fig. 1. We note that if $\alpha_0 \to 1$, the parameters of the mixture tend to the corresponding expressions for an ideal gas.

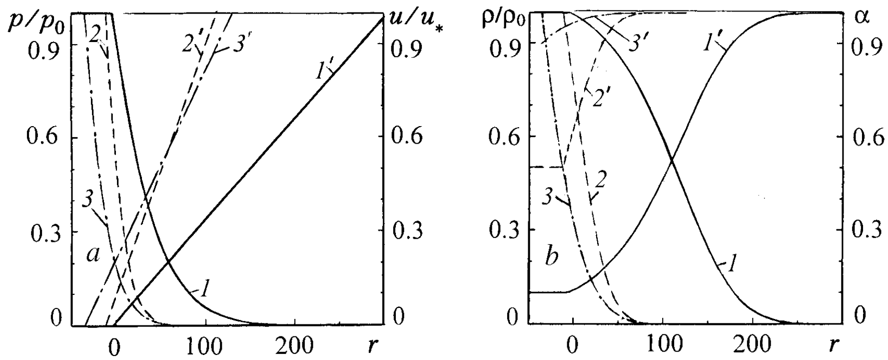

Fig. 2. Dependences $p/p_0(r)$ (1–3) and $u/u_*(r)$ (1′–3′) (a) and $\rho/\rho_0(r)$ (1–3) and $\alpha(r)$ (1′–3′) (b) in a water-air mixture at α_0 = 0.1, 0.5, and 0.9, respectively, for t = 20 msec.

From (4) and the third formula of (7) we have the nonlinear relation $\xi = \xi(u)$, the solution of which relative to u (which can be done, e.g., numerically) gives $u = u(\xi)$. The variable ξ changes from $-c_0$ to u_*, which follows from (4) provided that $c = c_0$ at $u = 0$ and $c = 0$ at $u = u_*$. Having substituted the dependence $u = u(\xi)$ into (6)–(7), we obtain the expressions of the sought functions of the self-similar variable ξ, i.e., $\rho = \rho(\xi)$, $p = p(\xi)$, $c = c(\xi)$, and $\alpha = \alpha(\xi)$. Fixing the time t in the relations obtained, we determine the instantaneous distributions $p = p(r)$, $u = u(r)$, $\rho = \rho(r)$, and $\alpha = \alpha(r)$ in space. On the contrary, fixing r, we obtain the dependences $p = p(t)$, $u = u(t)$, $\rho = \rho(t)$, and $\alpha = \alpha(t)$ at a given point of space. In particular, Fig. 2 presents the dependences of the sought parameters on the variable r for the instant of time t = 20 msec in the water-air mixture which were obtained in the calculations for different α_0 at an initial pressure of $p_0 = 10^5$ Pa (γ = 1.4, ρ_{10}^0 = 1.15 kg/m^3, and ρ_{20}^0 = 1000 kg/m^3). We note that as r increases the volume fraction of the gas in the mixture increases, whereas the true density of the gas decreases. As the boundary with vacuum is approached, the liquid fraction decreases.

We consider a mixture consisting of two compressible fractions which, for the sake of simplicity, are assumed to be ideal gases with adiabatic exponents γ_1 and γ_2. The determining equations have the form [1]

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\frac{\partial u}{\partial r}=0\,,\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0\,,$$

$$(A\alpha_1+B)\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial r}\right)+Ap\left(\frac{\partial\alpha_1}{\partial t}+u\frac{\partial\alpha_1}{\partial r}\right)+p\,(A\alpha_1+B+1)\frac{\partial u}{\partial r}=0\,,$$

$$\alpha_1\frac{\partial\rho_1^0}{\partial t}+\rho_1^0\frac{\partial\alpha_1}{\partial t}+\alpha_1 u\frac{\partial\rho_1^0}{\partial r}+\alpha_1\rho_1^0\frac{\partial u}{\partial r}+\rho_1^0 u\frac{\partial\alpha_1}{\partial r}=0\,,\tag{8}$$

$$\frac{\alpha_1}{\gamma_1-1}\left(\frac{\partial p}{\partial t}+\gamma_1 u\frac{\partial p}{\partial r}\right)+\alpha_1\rho_1^0 u\frac{\partial u}{\partial t}+\alpha_1\left(\frac{\gamma_1 p}{\gamma_1-1}+\frac{3\rho_1^0u^2}{2}\right)\frac{\partial u}{\partial r}+$$

$$+\frac{\alpha_1u^2}{2}\left(\frac{\partial\rho_1^0}{\partial t}+u\frac{\partial\rho_1^0}{\partial r}\right)+\left(\frac{p}{\gamma_1-1}+\frac{\rho_1^0u^2}{2}\right)\frac{\partial\alpha_1}{\partial t}+\left(\frac{\gamma_1 pu}{\gamma_1-1}+\frac{\rho_1^0u^3}{2}\right)\frac{\partial\alpha_1}{\partial r}=0\,,$$

where $A = (\gamma_2-\gamma_1)/((\gamma_1-1)(\gamma_2-1))$ and $B = 1/(\gamma_2-1)$. As previously, the first three equations of (8) are the laws of conservation of mass, momentum, and energy for the mixture as a whole, and the last two are the equations of continuity and energy for the first fraction. By introduction of the self-similar variable $\xi = r/t$

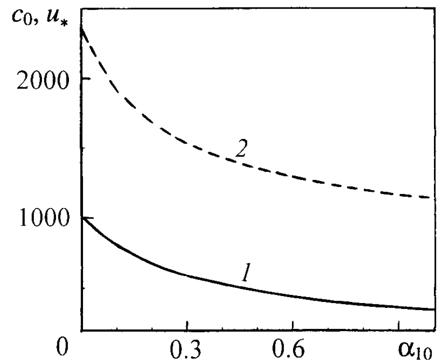

Fig. 3. Dependences $c_0(\alpha_{10})$ and $u_*(\alpha_{10})$ [1 and 2, respectively] for a nitrogen-helium mixture.

the system of partial differential equations (8) is, as previously, reduced to the system of ordinary differential equations

$$(u-\xi)\frac{d\rho}{d\xi}+\rho\frac{du}{d\xi}=0\,,\quad (u-\xi)\frac{du}{d\xi}+\frac{1}{\rho}\frac{dp}{d\xi}=0\,,$$

$$(A\alpha_1+B)(u-\xi)\frac{dp}{d\xi}+Ap\,(u-\xi)\frac{d\alpha_1}{d\xi}+p\,(A\alpha_1+B+1)\frac{du}{d\xi}=0\,,$$

$$\frac{\alpha_1}{\gamma_1-1}(\gamma_1 u-\xi)\frac{dp}{d\xi}+\alpha_1\left[\frac{\gamma_1 p}{\gamma_1-1}+u\rho_1^0\left(\frac{3u}{2}-\xi\right)\right]\frac{du}{d\xi}+$$

$$+\frac{\alpha_1 u^2(u-\xi)}{2}\frac{d\rho_1^0}{d\xi}+\left[\frac{p\,(\gamma_1 u-\xi)}{\gamma_1-1}+\frac{\rho_1^0 u^2(u-\xi)}{2}\right]\frac{d\alpha_1}{d\xi}=0\,, \tag{9}$$

$$\alpha_1\rho_1^0\frac{du}{d\xi}+\alpha_1(u-\xi)\frac{d\rho_1^0}{d\xi}+\rho_1^0(u-\xi)\frac{d\alpha_1}{d\xi}=0\,.$$

We note that in the considered case, the self-similar flow is isentropic. Eliminating $du/d\xi$ from the first two relations of (9), we obtain the equation $dp/d\rho=(u-\xi)^2=c^2$, whence

$$u-\xi=c\,, \tag{10}$$

where the velocity of sound is found from the formula [4]

$$c=\sqrt{\left(\rho_0\rho^{-2}\left[\frac{\alpha_{10}}{\gamma_1 p}\left(\frac{p_0}{p}\right)^{1/\gamma_1}+\frac{\alpha_{20}}{\gamma_2 p}\left(\frac{p_0}{p}\right)^{1/\gamma_2}\right]^{-1}\right)}. \tag{11}$$

Allowing for the fact that in the rarefaction wave the Riemann r-invariant is constant [4], we have

$$u=u\,(p)=\int_p^{p_0}\sqrt{\left(\frac{1}{\rho_0}\left[\frac{\alpha_{10}}{\gamma_1 p}\left(\frac{p_0}{p}\right)^{1/\gamma_1}+\frac{\alpha_{20}}{\gamma_2 p}\left(\frac{p_0}{p}\right)^{1/\gamma_2}\right]\right)}\,dp\,. \tag{12}$$

It is seen, in particular, from (12) that the velocity of the mixture reaches its maximum

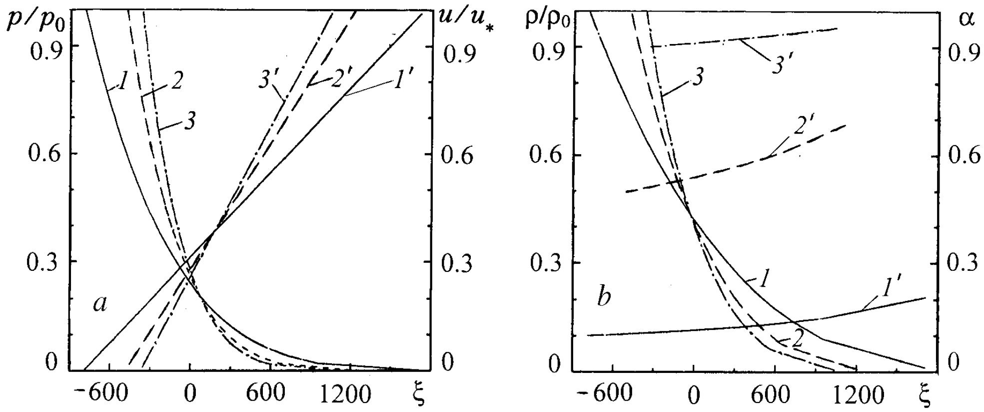

Fig. 4. Dependences $p/p_0(\xi)$ (1–3) and $u/u_*(\xi)$ (1′–3′) (a) and $\rho/\rho_0(\xi)$ (1–3) and $\alpha(\xi)$ (1′–3′) (b) in a nitrogen-helium mixture at α_{10} = 0.1, 0.5, and 0.9.

$$u_* = \int_0^{p_0} \sqrt{\left(\frac{1}{\rho_0}\left[\frac{\alpha_{10}}{\gamma_1 p}\left(\frac{p_0}{p}\right)^{1/\gamma_1} + \frac{\alpha_{20}}{\gamma_2 p}\left(\frac{p_0}{p}\right)^{1/\gamma_2}\right]\right)}\,dp \tag{13}$$

at $p = 0$ at the point near the boundary with vacuum. We note that integral (13), despite the presence of a singularity in the integrand at $p = 0$, converges to the limit. When, e.g., $\alpha_{10} = 1$, formula (13) gives the expression $u_* = 2c_0/(\gamma_1 - 1)$, which coincides with the corresponding dependence from single-phase gasdynamics.

Figure 3 presents the dependences of the velocities of motion of the rarefaction-wave boundaries in a nitrogen-helium mixture on the concentration of the nitrogen at $p_0 = 10^5$ Pa ($\gamma_1 = 1.4$, $\rho_{10}^0 = 1.15$ kg/m^3, $\gamma_2 = 1.67$, and $\rho_{20}^0 = 0.164$ kg/m^3). We note that, in contrast to the earlier considered case of a gas-liquid mixture with an incompressible liquid fraction, within the entire range of variation of α_{10} the dependences are finite. Since the flow in the rarefaction wave is isentropic, the relations from [1]

$$\rho = \frac{\rho_0}{\alpha_{10}\left(\frac{p_0}{p}\right)^{1/\gamma_1} + \alpha_{20}\left(\frac{p_0}{p}\right)^{1/\gamma_2}}, \quad \alpha_1 = \frac{\alpha_{10}\left(\frac{p_0}{p}\right)^{1/\gamma_1}}{\alpha_{10}\left(\frac{p_0}{p}\right)^{1/\gamma_1} + \alpha_{20}\left(\frac{p_0}{p}\right)^{1/\gamma_2}} \tag{14}$$

hold, which in combination with Eqs. (10)–(12) allow determination of the dependences $p = p(\xi)$, $u = u(\xi)$, $\rho = \rho(\xi)$, and $\alpha_1 = \alpha_1(\xi)$. We find the dependence $\rho_1^0 = \rho_1^0(\xi)$ (still unknown) from the differential equation

$$\frac{d\rho_1^0}{d\xi} = -\frac{2}{\alpha_1 u^2 (u-\xi)}\left[\frac{\gamma_1 p - \rho (u-\xi)^2 (A\alpha_1 + B)}{\gamma_1 - 1} + \rho_1^0 u\left(\frac{3u}{2} - \xi\right) - \right.$$
$$\left. - \frac{p (A\alpha_1 + B + 1) - \rho (u-\xi)^2 (A\alpha_1 + B)}{Ap (u-\xi)}\left(\frac{p (\gamma_1 u - \xi)}{\gamma_1 - 1} + \frac{\rho_1^0 u^2 (u-\xi)}{2}\right)\right]\frac{du}{d\xi}, \tag{15}$$

which follows from system (9).

Figure 4 gives the dependences of the distributions of the sought parameters on the self-similar variable ξ which are obtained in the calculations for a nitrogen-helium mixture at different α_{10}.

The outflow of a mixture with a large number of components into vacuum is considered quite similarly; in this case, the number of differential equations increases, which, however, only leads to more cumbersome calculations. The calculational technique remains the same. We also note that the rarefaction flow in the general case where the pressure to the right of the orifice differs from zero is calculated by analogy with outflow into vacuum. This makes it possible, in particular, to supplement [5], where the Riemann problem has been solved but the distributions of parameters in the rarefaction wave have not been determined.

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NOTATION

p, pressure, Pa; u, velocity, m/sec; γ, adiabatic exponent; S, entropy; c, velocity of sound, m/sec; r, distance, m; t, time, sec; ρ, mixture density, kg/m^3; ρ_i^0, true density of the ith fraction, kg/m^3; α_i, volume fraction of the ith component of the mixture. Super- and subscripts: 0, initial state; *, critical value.

REFERENCES

1. V. S. Surov, *Teplofiz. Vys. Temp.*, **38**, No. 1, 165 (2000).
2. V. S. Surov, *Mat. Model.*, **10**, No. 3, 29–38 (1998).
3. V. S. Surov, *Teplofiz. Vys. Temp.*, **36**, No. 2, 272–278 (1998).
4. V. S. Surov, *Teplofiz. Vys. Temp.*, **36**, No. 4, 624–630 (1998).
5. V. S. Surov, *Teplofiz. Vys. Temp.*, **36**, No. 1, 157–161 (1998).